

# Dimer-monomer model on the Sierpinski gasket

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## Abstract

We present the numbers of dimer-monomers on the Sierpinski gasket  $SG_d(n)$  at stage  $n$  with dimension  $d$  equal to two, three and four, and determine the asymptotic behaviors. The corresponding results on the generalized Sierpinski gasket  $SG_{d,b}(n)$  with  $d = 2$  and  $b = 3, 4$  are obtained.

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## I. INTRODUCTION

The enumeration of the number of dimer-monomers  $N_{DM}(G)$  on a graph  $G$  is a classical model [1, 2, 3]. In the model, each diatomic molecule is regarded as a dimer which occupies two adjacent sites of the graph. The sites that are not covered by any dimers are considered as occupied by monomers. Although the close-packed dimer problem on planar lattices has been expressed in closed-form almost half a century ago [4, 5, 6], the general dimer-monomer problem was shown to be computationally intractable [7]. Some recent studies on the enumeration of close-packed dimer, single-monomer and general dimer-monomer problems on regular lattices were carried out in Refs. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. It is of interest to consider dimer-monomers on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of non-integer Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [19, 20]. A well-known example of fractal is the Sierpinski gasket which has been extensively studied in several contexts [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. We shall derive the recursion relations for the numbers of dimer-monomers on the Sierpinski gasket with dimension equal to two, three and four, and determine the asymptotic growth constants. We shall also consider the number of dimer-monomers on the generalized Sierpinski gasket with dimension equal to two.

## II. PRELIMINARIES

We first recall some relevant definitions in this section. A connected graph (without loops)  $G = (V, E)$  is defined by its vertex (site) and edge (bond) sets  $V$  and  $E$  [32, 33]. Let  $v(G) = |V|$  be the number of vertices and  $e(G) = |E|$  the number of edges in  $G$ . The degree or coordination number  $k_i$  of a vertex  $v_i \in V$  is the number of edges attached to it. A  $k$ -regular graph is a graph with the property that each of its vertices has the same degree  $k$ . In general, one can associate monomer and dimer weights to each monomer and dimer connecting adjacent vertices (see, for example [16]). For simplicity, all such weights are set to one throughout this paper.

When the number of dimer-monomers  $N_{DM}(G)$  grows exponentially with  $v(G)$  as  $v(G) \rightarrow$

$\infty$ , there exists a constant  $z_G$  describing this exponential growth:

$$z_G = \lim_{v(G) \rightarrow \infty} \frac{\ln N_{DM}(G)}{v(G)}, \quad (2.1)$$

where  $G$ , when used as a subscript in this manner, implicitly refers to the thermodynamic limit.

The construction of the two-dimensional Sierpinski gasket  $SG_2(n)$  at stage  $n$  is shown in Fig. 1. At stage  $n = 0$ , it is an equilateral triangle; while stage  $n + 1$  is obtained by the juxtaposition of three  $n$ -stage structures. In general, the Sierpinski gaskets  $SG_d$  can be built in any Euclidean dimension  $d$  with fractal dimensionality  $D = \ln(d + 1)/\ln 2$  [22]. For the Sierpinski gasket  $SG_d(n)$ , the numbers of edges and vertices are given by

$$e(SG_d(n)) = \binom{d+1}{2} (d+1)^n = \frac{d}{2} (d+1)^{n+1}, \quad (2.2)$$

$$v(SG_d(n)) = \frac{d+1}{2} [(d+1)^n + 1]. \quad (2.3)$$

Except the  $(d + 1)$  outmost vertices which have degree  $d$ , all other vertices of  $SG_d(n)$  have degree  $2d$ . In the large  $n$  limit,  $SG_d$  is  $2d$ -regular.

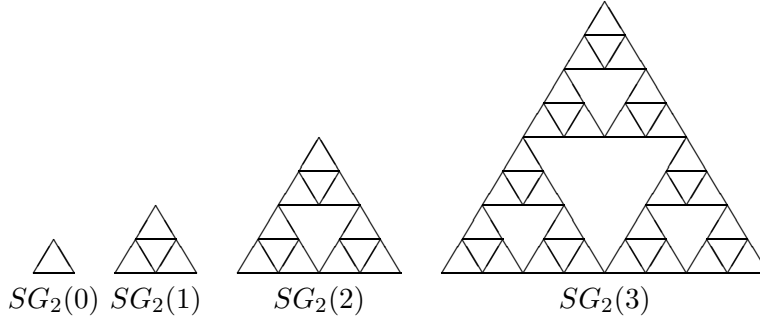


FIG. 1: The first four stages  $n = 0, 1, 2, 3$  of the two-dimensional Sierpinski gasket  $SG_2(n)$ .

The Sierpinski gasket can be generalized, denoted as  $SG_{d,b}(n)$ , by introducing the side length  $b$  which is an integer larger or equal to two [34]. The generalized Sierpinski gasket at stage  $n + 1$  is constructed with  $b$  layers of stage  $n$  hypertetrahedrons. The two-dimensional  $SG_{2,b}(n)$  with  $b = 3$  at stage  $n = 1, 2$  and  $b = 4$  at stage  $n = 1$  are illustrated in Fig. 2. The ordinary Sierpinski gasket  $SG_d(n)$  corresponds to the  $b = 2$  case, where the index  $b$  is neglected for simplicity. The Hausdorff dimension for  $SG_{d,b}$  is given by  $D = \ln \binom{b+d-1}{d} / \ln b$  [34]. Notice that  $SG_{d,b}$  is not  $k$ -regular even in the thermodynamic limit.

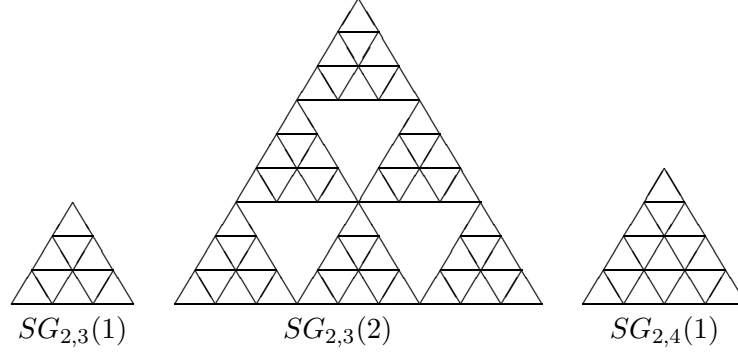


FIG. 2: The generalized two-dimensional Sierpinski gasket  $SG_{2,b}(n)$  with  $b = 3$  at stage  $n = 1, 2$  and  $b = 4$  at stage  $n = 1$ .

### III. THE NUMBER OF DIMER-MONOMERS ON $SG_2(n)$

In this section we derive the asymptotic growth constant for the number of dimer-monomers on the two-dimensional Sierpinski gasket  $SG_2(n)$  in detail. Let us start with the definitions of the quantities to be used.

**Definition III.1** Consider the generalized two-dimensional Sierpinski gasket  $SG_{2,b}(n)$  at stage  $n$ . (a) Define  $M_{2,b}(n) \equiv N_{DM}(SG_{2,b}(n))$  as the number of dimer-monomers. (b) Define  $f_{2,b}(n)$  as the number of dimer-monomers such that the three outmost vertices are occupied by monomers. (c) Define  $gr_{2,b}(n)$ ,  $gl_{2,b}(n)$ ,  $gt_{2,b}(n)$  as the numbers of dimer-monomers such that either rightmost, leftmost or topmost vertex, respectively, is occupied by a dimer and the other two outmost vertices are occupied by monomers. (d) Define  $hr_{2,b}(n)$ ,  $hl_{2,b}(n)$ ,  $ht_{2,b}(n)$  as the numbers of dimer-monomers such that either rightmost, leftmost or topmost vertex, respectively, is occupied by a monomer and the other two outmost vertices are occupied by dimers. (e) Define  $t_{2,b}(n)$  as the number of dimer-monomers such that all three outmost vertices are occupied by dimers.

It is clear that the values  $gr_{2,b}(n)$ ,  $gl_{2,b}(n)$ ,  $gt_{2,b}(n)$  are the same because of rotation symmetry, and we define  $g_{2,b}(n) \equiv gr_{2,b}(n) = gl_{2,b}(n) = gt_{2,b}(n)$ . Similarly, we define  $h_{2,b}(n) \equiv hr_{2,b}(n) = hl_{2,b}(n) = ht_{2,b}(n)$ . Since we only consider ordinary Sierpinski gasket in this section, we use the notations  $M_2(n)$ ,  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$ , and  $t_2(n)$  for simplicity. They are illustrated in Fig. 3, where only the outmost vertices are shown. It follows that

$$M_2(n) = f_2(n) + 3g_2(n) + 3h_2(n) + t_2(n) \quad (3.1)$$

for non-negative integer  $n$ . The initial values at stage zero are  $f_2(0) = 1$ ,  $g_2(0) = 0$ ,  $h_2(0) = 1$ ,  $t_2(0) = 0$  and  $M_2(0) = 4$ . The values at stage one are  $f_2(1) = 4$ ,  $g_2(1) = 4$ ,  $h_2(1) = 3$ ,  $t_2(1) = 2$  and  $M_2(1) = 27$ . The purpose of this section is to obtain the asymptotic behavior of  $M_2(n)$  as follows. The five quantities  $M_2(n)$ ,  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$  and  $t_2(n)$  satisfy recursion relations.

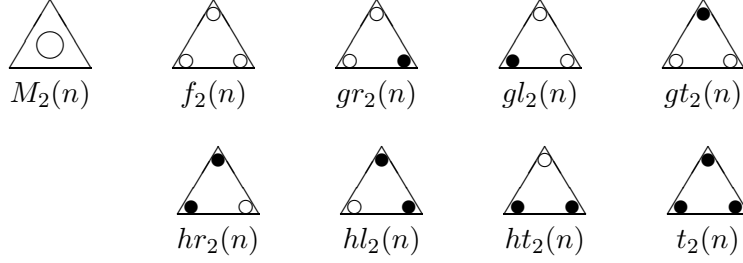


FIG. 3: Illustration for the configurations  $M_2(n)$ ,  $f_2(n)$ ,  $gr_2(n)$ ,  $gl_2(n)$ ,  $gt_2(n)$ ,  $hr_2(n)$ ,  $hl_2(n)$ ,  $ht_2(n)$  and  $t_2(n)$ . Only the three outmost vertices are shown explicitly for  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$  and  $t_2(n)$ , where each open circle is occupied by a monomer and each solid circle is occupied by a dimer.

**Lemma III.1** *For any non-negative integer  $n$ ,*

$$M_2(n+1) = M_2^3(n) - 3M_2(n)[g_2(n) + 2h_2(n) + t_2(n)]^2 + 3[h_2(n) + t_2(n)][g_2(n) + 2h_2(n) + t_2(n)]^2 - [h_2(n) + t_2(n)]^3, \quad (3.2)$$

$$f_2(n+1) = f_2^3(n) + 6f_2^2(n)g_2(n) + 3f_2^2(n)h_2(n) + 9f_2(n)g_2^2(n) + 2g_2^3(n) + 6f_2(n)g_2(n)h_2(n), \quad (3.3)$$

$$g_2(n+1) = f_2^2(n)g_2(n) + 2f_2^2(n)h_2(n) + 4f_2(n)g_2^2(n) + f_2^2(n)t_2(n) + 8f_2(n)g_2(n)h_2(n) + 3g_2^3(n) + 2f_2(n)g_2(n)t_2(n) + 2f_2(n)h_2^2(n) + 4g_2^2(n)h_2(n), \quad (3.4)$$

$$h_2(n+1) = f_2(n)g_2^2(n) + 4f_2(n)g_2(n)h_2(n) + 2g_2^3(n) + 2f_2(n)g_2(n)t_2(n) + 7g_2^2(n)h_2(n)$$

$$+3f_2(n)h_2^2(n) + 2f_2(n)h_2(n)t_2(n) + 2g_2^2(n)t_2(n) + 4g_2(n)h_2^2(n) , \quad (3.5)$$

$$t_2(n+1) = g_2^3(n) + 6g_2^2(n)h_2(n) + 3g_2^2(n)t_2(n) + 9g_2(n)h_2^2(n) + 2h_2^3(n) \\ + 6g_2(n)h_2(n)t_2(n) . \quad (3.6)$$

*Proof* The Sierpinski gaskets  $SG_2(n+1)$  is composed of three  $SG_2(n)$  with three pairs of vertices identified. For the number  $M_2(n+1)$ , the unallowable configurations are those with at least a pair of identified vertices originally occupied by dimers. Therefore, the three configuration with a pair of identified vertices occupied by dimers should be subtracted from all possible configurations  $M_2^3(n)$ . However, this procedure subtracts out also configurations with two pairs of identified vertices occupied by dimers that should be added back as illustrated in Fig. 4. Finally, the configuration with three pairs of identified vertices occupied by dimers should be subtracted, and Eq. (3.2) is verified.

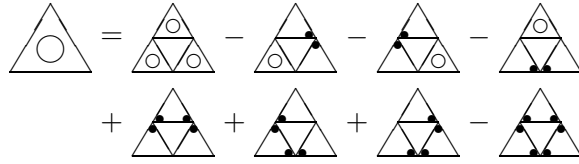


FIG. 4: Illustration for the expression of  $M_2(n+1)$ . Certain outmost vertices of  $SG_2(n)$  are not shown means they can be occupied by either dimers or monomers.

As illustrated in Fig. 5, the number  $f_2(n+1)$  consists of (i) one configuration where all three of the  $SG_2(n)$  are in the  $f_2(n)$  status, (ii) six configurations where two of the  $SG_2(n)$  are in the  $f_2(n)$  status and the other one is in the  $g_2(n)$  status, (iii) three configurations where two of the  $SG_2(n)$  are in the  $f_2(n)$  status and the other one is in the  $h_2(n)$  status, (iv) nine configurations where one of the  $SG_2(n)$  is in the  $f_2(n)$  status and the other two are in the  $g_2(n)$  status, (v) two configuration where all three of the  $SG_2(n)$  are in the  $g_2(n)$  status, (vi) six configurations where one of the  $SG_2(n)$  is in the  $f_2(n)$  status, another one is in the  $g_2(n)$  status and the other one is in the  $h_2(n)$  status. Therefore, we have

$$f_2(n+1) = f_2^3(n) + 2f_2^2(n)[gr_2(n) + gl_2(n) + gt_2(n)] + f_2^2(n)[hr_2(n) + hl_2(n) + ht_2(n)]$$

$$\begin{aligned}
& +2f_2(n)[gt_2(n)gr_2(n) + gt_2(n)gl_2(n) + gr_2(n)gl_2(n)] \\
& +f_2(n)[gr_2^2(n) + gl_2^2(n) + gt_2^2(n)] + 2gr_2(n)gl_2(n)gt_2(n) \\
& +f_2(n)[gt_2(n)hr_2(n) + gl_2(n)ht_2(n) + gr_2(n)hl_2(n)] \\
& +f_2(n)[ht_2(n)gr_2(n) + hl_2(n)gt_2(n) + hr_2(n)gl_2(n)] . \tag{3.7}
\end{aligned}$$

With the identity  $gr_2(n) = gl_2(n) = gt_2(n) = g_2(n)$  and  $hr_2(n) = hl_2(n) = ht_2(n) = h_2(n)$ , Eq. (3.3) is verified.

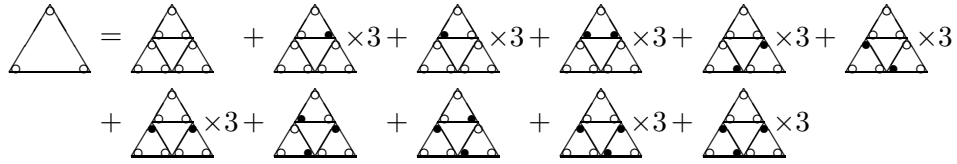


FIG. 5: Illustration for the expression of  $f_2(n+1)$ . The multiplication of three on the right-hand-side corresponds to the three possible orientations of  $SG_2(n+1)$ .

Similarly,  $gt_2(n+1) = g_2(n+1)$ ,  $ht_2(n+1) = h_2(n+1)$  and  $t_2(n+1)$  for  $SG_2(n+1)$  can be obtained with appropriate configurations of its three constituting  $SG_2(n)$  as illustrated in Figs. 6, 7 and 8 to verify Eqs. (3.4), (3.5) and (3.6), respectively.

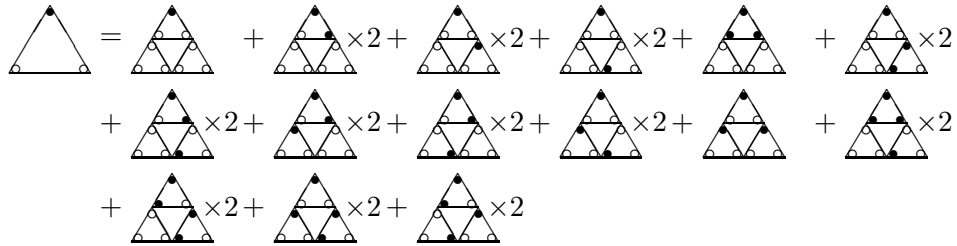


FIG. 6: Illustration for the expression of  $gt_2(n+1)$ . The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

Eq. (3.2) can also be obtained by substituting Eqs. (3.3)-(3.6) into Eq. (3.1).  $\square$

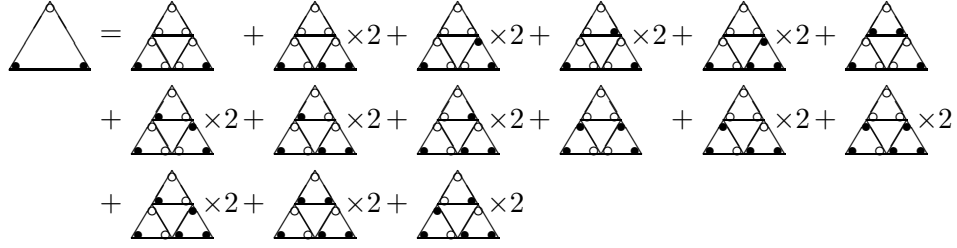


FIG. 7: Illustration for the expression of  $ht_2(n+1)$ . The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

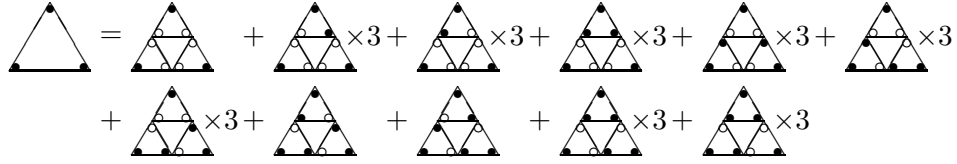


FIG. 8: Illustration for the expression of  $t_2(n+1)$ . The multiplication of three on the right-hand-side corresponds to the three possible orientations of  $SG_2(n+1)$ .

There are always  $27 = 3^3$  terms in Eqs. (3.3), (3.4), (3.5), (3.6) because there are three possible choices for each of the three pairs of identified vertices: both of them are originally occupied by monomers, or either one of them is originally occupied by a monomer while the other one by a dimer. The values of  $M_2(n)$ ,  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$ ,  $t_2(n)$  for small  $n$  can be evaluated recursively by Eqs. (3.2)-(3.6) as listed in Table I. These numbers grow exponentially, and do not have simple integer factorizations. To estimate the value of the asymptotic growth constant defined in Eq. (2.1), we need the following lemmas. For the generalized two-dimensional Sierpinski gasket  $SG_{2,b}(n)$ , define the ratios

$$\alpha_{2,b}(n) = \frac{g_{2,b}(n)}{f_{2,b}(n)}, \quad \beta_{2,b}(n) = \frac{h_{2,b}(n)}{g_{2,b}(n)}, \quad \gamma_{2,b}(n) = \frac{t_{2,b}(n)}{h_{2,b}(n)}, \quad (3.8)$$

For the ordinary Sierpinski gasket in this section, they are simplified to be  $\alpha_2(n)$ ,  $\beta_2(n)$  and  $\gamma_2(n)$ .

**Lemma III.2** *For any positive integer  $n$ , the magnitudes of  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$ ,  $t_2(n)$  are ordered as*

$$t_2(n) \leq h_2(n) \leq g_2(n) \leq f_2(n), \quad (3.9)$$



TABLE I: The first few values of  $M_2(n)$ ,  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$ ,  $t_2(n)$ .

$n$	0	1	2	3	4
$M_2(n)$	4	27	10,054	499,058,851,840	60,978,122,299,433,248,924,629,725,740,007,424
$f_2(n)$	1	4	1,584	78,721,368,064	9,618,673,427,679,675,357,952,788,786,053,120
$g_2(n)$	0	4	1,352	66,974,056,448	8,183,299,472,241,085,511,976,093,040,508,928
$h_2(n)$	1	3	1,148	56,979,607,552	6,962,123,286,110,084,944,276,569,997,705,216
$t_2(n)$	0	2	970	48,476,491,776	5,923,180,596,700,062,197,918,947,839,311,872

then

$$0 \leq \gamma_2(n) \leq \beta_2(n) \leq \alpha_2(n) \leq 1 . \quad (3.10)$$

*Proof* Eq. (3.9) is valid for the first few positive integer  $n$  by the numbers given in Table I. By Eqs. (3.3)-(3.6), we have

$$\begin{aligned} \frac{f_2(n+1)}{f_2^3(n)} - \frac{g_2(n+1)}{f_2^2(n)g_2(n)} &= [\alpha_2(n) - \beta_2(n)][2 + 5\alpha_2(n) + 2\alpha_2(n)\beta_2(n)] \\ &\quad + [1 + 2\alpha_2(n)][\alpha_2^2(n) - \beta_2(n)\gamma_2(n)] , \end{aligned} \quad (3.11)$$

$$\begin{aligned} \frac{g_2(n+1)}{f_2^2(n)g_2(n)} - \frac{h_2(n+1)}{f_2(n)g_2^2(n)} &= [\alpha_2(n) - \beta_2(n)][2 + 3\alpha_2(n) + 3\beta_2(n) + 2\alpha_2(n)\beta_2(n)] \\ &\quad + \beta_2(n)[\alpha_2(n) - \gamma_2(n) + 2[\alpha_2^2(n) - \beta_2(n)\gamma_2(n)]] , \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{h_2(n+1)}{f_2(n)g_2^2(n)} - \frac{t_2(n+1)}{g_2^3(n)} &= [\alpha_2(n) - \beta_2(n)][2 + 6\beta_2(n) + 2\beta_2^2(n) + 2\beta_2(n)\gamma_2(n)] \\ &\quad + \beta_2(n)[\alpha_2(n) - \gamma_2(n)][1 + 2\beta_2(n)] . \end{aligned} \quad (3.13)$$

Eq. (3.9) is proved by mathematical induction if Eqs. (3.11)-(3.13) are larger or equal to zero. Equivalently, we need  $\gamma_2(n) \leq \beta_2(n) \leq \alpha_2(n)$ , which can be proved also by induction since the following two expressions are larger or equal to zero for any positive integer  $n$ .

$$\left[ \frac{g_2(n+1)}{f_2^2(n)g_2(n)} \right]^2 - \frac{f_2(n+1)}{f_2^3(n)} \frac{h_2(n+1)}{f_2(n)g_2^2(n)}$$

$$\begin{aligned}
&= [\alpha_2(n) - \beta_2(n)]^2[1 + 3\alpha_2(n) + \gamma_2(n) + 4\alpha_2^2(n)[\beta_2(n) + \gamma_2(n)] + 4\alpha_2^2(n)\beta_2^2(n)] \\
&\quad + [\alpha_2^2(n) - \beta_2^2(n)][[\alpha_2(n) - \gamma_2(n)][1 + 2\alpha_2(n)\beta_2(n)] + 2\alpha_2^2(n)[\beta_2^2(n) - \gamma_2^2(n)]] \\
&\quad + [\alpha_2(n) - \beta_2(n)][4\alpha_2(n)[\alpha_2^2(n) - \beta_2(n)\gamma_2(n)] + \alpha_2(n)\beta_2(n)[\alpha_2(n) - \gamma_2(n)] \\
&\quad + \alpha_2(n)\beta_2(n)[\beta_2(n) - \gamma_2(n)]] + 4\alpha_2(n)[\alpha_2^3(n) - \beta_2^3(n)][\beta_2(n) - \gamma_2(n)] \\
&\quad + [\alpha_2^2(n) - \beta_2(n)\gamma_2(n)]^2 + 2\alpha_2(n)[\beta_2(n) - \gamma_2(n)]^2[2\beta_2^2(n) + \alpha_2^3(n) + \alpha_2(n)\beta_2^2(n)] ,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\left[ \frac{h_2(n+1)}{f_2(n)g_2^2(n)} \right]^2 - \frac{g_2(n+1)}{f_2^2(n)g_2(n)} \frac{t_2(n+1)}{g_2^3(n)} \\
&= [\alpha_2(n) - \beta_2(n)]^2[1 + 4\beta_2(n) + 2\gamma_2(n) + 9\beta_2^2(n) + 6\beta_2^3(n) + 6\beta_2^2(n)\gamma_2(n) + 2\beta_2^2(n)\gamma_2^2(n)] \\
&\quad + [\alpha_2(n) - \beta_2(n)][2\alpha_2(n)[\beta_2(n) - \gamma_2(n)] + 4\beta_2^2(n)[\alpha_2(n) + \beta_2(n) - 2\gamma_2(n)] \\
&\quad + 2\alpha_2(n)\beta_2^2(n)[\beta_2^2(n) - \gamma_2^2(n)] + 2\beta_2^3(n)[\alpha_2(n)\beta_2(n) - \gamma_2^2(n)]] \\
&\quad + \beta_2(n)[\alpha_2^2(n) - \beta_2^2(n)][\beta_2(n) - \gamma_2(n)][1 + 8\beta_2(n)] \\
&\quad + \beta_2^2(n)[\beta_2(n) - \gamma_2(n)]^2[1 + 2\alpha_2(n) + 2\beta_2(n) + 4\alpha_2^2(n)] .
\end{aligned} \tag{3.15}$$

It is clear that  $0 \leq \gamma_2(n)$  since all the quantities  $f_2(n)$ ,  $g_2(n)$ ,  $h_2(n)$ ,  $t_2(n)$  are positive. Finally,  $\alpha_2(n) \leq 1$  once Eq. (3.9) is established.  $\square$

The values of  $\alpha_2(n)$ ,  $\beta_2(n)$ ,  $\gamma_2(n)$  for small  $n$  are listed in Table II.

**Lemma III.3** *Sequence  $\{\alpha_2(n)\}_{n=1}^{\infty}$  decreases monotonically, while sequences  $\{\gamma_2(n)\}_{n=1}^{\infty}$  and  $\{t_2(n)/f_2(n)\}_{n=1}^{\infty}$  increase monotonically. The limits  $\alpha_2 \equiv \lim_{n \rightarrow \infty} \alpha_2(n)$ ,  $\beta_2 \equiv \lim_{n \rightarrow \infty} \beta_2(n)$ ,  $\gamma_2 \equiv \lim_{n \rightarrow \infty} \gamma_2(n)$  exist.*

TABLE II: The first few values of  $\alpha_2(n)$ ,  $\beta_2(n)$ ,  $\gamma_2(n)$ . The last digits given are rounded off.

$n$	1	2	3	4
$\alpha_2(n)$	1	0.853535353535354	0.850773533223540	0.850772150002722
$\beta_2(n)$	0.75	0.849112426035503	0.850771337051088	0.850772150002159
$\gamma_2(n)$	0.666666666666667	0.844947735191638	0.850769141078411	0.850772150001597

*Proof* By Eqs. (3.3) and (3.4), we have

$$\begin{aligned}
& f_2(n+1)g_2(n) - g_2(n+1)f_2(n) \\
&= f_2^2(n)g_2(n) \left[ [2f_2(n) + 5g_2(n) + 2h_2(n)][\alpha_2(n) - \beta_2(n)] \right. \\
&\quad \left. + [f_2(n) + 2g_2(n)] \left[ \frac{g_2^2(n)}{f_2^2(n)} - \frac{t_2(n)}{g_2(n)} \right] \right] \geq 0 .
\end{aligned} \tag{3.16}$$

By induction,  $\alpha_2(n)$  decreases as positive  $n$  increases using the results of Lemma III.2. By Eqs. (3.5) and (3.6), we have

$$\begin{aligned}
& t_2(n+1)h_2(n) - h_2(n+1)t_2(n) \\
&= f_2(n)h_2(n)[g_2^2(n) + 4g_2(n)h_2(n) + 3h_2^2(n)][\alpha_2(n) - \gamma_2(n)] \\
&\quad + 2g_2^2(n)h_2(n)[g_2(n) + 2h_2(n) + t_2(n)][\beta_2(n) - \gamma_2(n)] \\
&\quad + 2f_2(n)h_2^2(n)[g_2(n) + h_2(n)] \left[ \frac{h_2(n)}{f_2(n)} - \frac{t_2^2(n)}{h_2^2(n)} \right] \geq 0 ,
\end{aligned} \tag{3.17}$$

such that  $\gamma_2(n)$  increases as positive  $n$  increases. Finally, we have

$$\begin{aligned}
& t_2(n+1)f_2(n) - f_2(n+1)t_2(n) \\
&= f_2^4(n) \left[ \frac{g_2^3(n)}{f_2^3(n)} - \frac{t_2(n)}{f_2(n)} \right] + 6f_2^2(n)g_2(n)h_2(n)[\alpha_2(n) - \gamma_2(n)] \\
&\quad + 3f_2^2(n)g_2(n)t_2(n)[\alpha_2(n) - \beta_2(n)] + 9f_2(n)g_2^2(n)h_2(n)[\beta_2(n) - \gamma_2(n)]
\end{aligned}$$

$$-2f_2(n)g_2^2(n)t_2(n)\left[\frac{g_2(n)}{f_2(n)} - \frac{h_2^3(n)}{g_2^2(n)t_2(n)}\right] \geq 0, \quad (3.18)$$

where the last inequality holds because of the combination of the third and the last terms:

$$2f_2(n)g_2(n)t_2(n)\left[f_2(n)\left[\frac{g_2(n)}{f_2(n)} - \frac{h_2(n)}{g_2(n)}\right] - g_2(n)\left[\frac{g_2(n)}{f_2(n)} - \frac{h_2^3(n)}{g_2^2(n)t_2(n)}\right]\right] \geq 0. \quad (3.19)$$

Because the sequence  $\alpha_2(n)$  decreases monotonically and bounded below, the limit  $\alpha_2$  exists. Similarly, sequence  $\gamma_2(n)$  and  $t_2(n)/f_2(n)$  increases monotonically and bounded above so that the limits  $\gamma_2$  and  $\lim_{n \rightarrow \infty} t_2(n)/f_2(n)$  exist. It follows that the limit  $\lim_{n \rightarrow \infty} h_2(n)/f_2(n)$  exists since

$$\lim_{n \rightarrow \infty} \frac{t_2(n)}{f_2(n)} = \lim_{n \rightarrow \infty} \frac{h_2(n)}{f_2(n)} \lim_{n \rightarrow \infty} \frac{t_2(n)}{h_2(n)}, \quad (3.20)$$

such that  $\beta = \lim_{n \rightarrow \infty} h_2(n)/g_2(n)$  exists because

$$\lim_{n \rightarrow \infty} \frac{h_2(n)}{f_2(n)} = \lim_{n \rightarrow \infty} \frac{g_2(n)}{f_2(n)} \lim_{n \rightarrow \infty} \frac{h_2(n)}{g_2(n)}. \quad (3.21)$$

□

With the existence of the limits  $\alpha_2$ ,  $\beta_2$ ,  $\gamma_2$ , and  $\gamma_2 \leq \beta_2 \leq \alpha_2$ , we have

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{f_2(n+1)}{f_2(n)} \frac{g_2(n)}{g_2(n+1)} \\ &= \frac{(1 + 3\alpha_2)^2 + 2\alpha_2^3 + 3\alpha_2\beta_2(1 + 2\alpha_2)}{1 + 2\beta_2 + 4\alpha_2 + \beta_2\gamma_2 + 8\alpha_2\beta_2 + 3\alpha_2^2 + 2\alpha_2\beta_2\gamma_2 + 2\alpha_2\beta_2^2 + 4\alpha_2^2\beta_2} \end{aligned} \quad (3.22)$$

by Eqs. (3.3) and (3.4), which leads to the following result.

**Corollary III.1** *The three limits  $\gamma_2$ ,  $\beta_2$  and  $\alpha_2$  are equal to each other.*

**Lemma III.4** *The asymptotic growth constant for the number of dimer-monomers on  $SG_2(n)$  is bounded:*

$$\frac{2}{3^{m+1}} \ln f_2(m) + \frac{\ln[1 + 2\gamma_2(m)]}{3^m} \leq z_{SG_2} \leq \frac{2}{3^{m+1}} \ln f_2(m) + \frac{\ln[1 + 2\alpha_2(m)]}{3^m}, \quad (3.23)$$

where  $m$  is a positive integer.

*Proof* Let us define  $\lambda_2(n) = f_2(n+1)/f_2^3(n)$ . By Eq. (3.3), we have

$$\lambda_2(n) = [1 + 3\alpha_2(n)]^2 + 2\alpha_2^3(n) + 3\alpha_2(n)\beta_2(n)[1 + 2\alpha_2(n)] . \quad (3.24)$$

It is clear that  $1 \leq \lambda_n \leq 27$ , and

$$[1 + 2\gamma_2(m)]^3 \leq [1 + 2\gamma_2(n)]^3 \leq [1 + 2\beta_2(n)]^3 \leq \lambda_2(n) \leq [1 + 2\alpha_2(n)]^3 \leq [1 + 2\alpha_2(m)]^3 \quad (3.25)$$

for  $n \geq m$ . By Eqs. (2.3) and (3.1), we have

$$\frac{\ln M_2(n)}{v(SG_2(n))} = \frac{2 \ln[1 + 3\alpha_2(n) + 3\alpha_2(n)\beta_2(n) + \alpha_2(n)\beta_2(n)\gamma_2(n)]}{3(3^n + 1)} + \frac{2 \ln f_2(n)}{3(3^n + 1)} , \quad (3.26)$$

where

$$\begin{aligned} \ln f_2(n) &= \ln \lambda_2(n-1) + 3 \ln f_2(n-1) \\ &= \ln \lambda_2(n-1) + 3 \ln \lambda_2(n-2) + 3^2 \ln f_2(n-2) \\ &= \dots \\ &= \sum_{j=m}^{n-1} 3^{n-1-j} \ln \lambda_2(j) + 3^{n-m} \ln f_2(m) \end{aligned} \quad (3.27)$$

for any  $m < n$ . By the definition of the asymptotic growth constant in Eq. (2.1),

$$\begin{aligned} z_{SG_2} &= \lim_{n \rightarrow \infty} \frac{\ln M_2(n)}{v(SG_2(n))} \\ &= \lim_{n \rightarrow \infty} \frac{2 \ln[1 + 3\alpha_2(n) + 3\alpha_2(n)\beta_2(n) + \alpha_2(n)\beta_2(n)\gamma_2(n)]}{3(3^n + 1)} \\ &\quad + \lim_{n \rightarrow \infty} \frac{2 \sum_{j=m}^{n-1} 3^{n-1-j} \ln \lambda_2(j) + 2[3^{n-m} \ln f_2(m)]}{3(3^n + 1)} \\ &= \frac{2}{3^2} \sum_{j=m}^{\infty} \frac{\ln \lambda_2(j)}{3^j} + \frac{2}{3^{m+1}} \ln f_2(m) . \end{aligned} \quad (3.28)$$

The proof is completed using the inequality (3.25).  $\square$

The difference between the upper and lower bounds for  $z_{SG_2}$  quickly converges to zero as  $m$  increases, and we have the following proposition.

**Proposition III.1** *The asymptotic growth constant for the number of dimer-monomers on the two-dimensional Sierpinski gasket  $SG_2(n)$  in the large  $n$  limit is  $z_{SG_2} = 0.656294236916\dots$*

The numerical value of  $z_{SG_2}$  can be calculated with more than a hundred significant figures accurate when  $m$  in Eq. (3.23) is equal to seven.

#### IV. THE NUMBER OF DIMER-MONOMERS ON $SG_{2,b}(n)$ WITH $b = 3, 4$

The method given in the previous section can be applied to the number of dimer-monomers on  $SG_{d,b}(n)$  with larger values of  $d$  and  $b$ . The number of configurations to be considered increases as  $d$  and  $b$  increase, and the recursion relations must be derived individually for each  $d$  and  $b$ . In this section, we consider the generalized two-dimensional Sierpinski gasket  $SG_{2,b}(n)$  with the number of layers  $b$  equal to three and four. For  $SG_{2,3}(n)$ , the numbers of edges and vertices are given by

$$e(SG_{2,3}(n)) = 3 \times 6^n, \quad (4.1)$$

$$v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5}, \quad (4.2)$$

where the three outmost vertices have degree two. There are  $(6^n - 1)/5$  vertices of  $SG_{2,3}(n)$  with degree six and  $6(6^n - 1)/5$  vertices with degree four. By Definition III.1, the number of dimer-monomers is  $M_{2,3}(n) = f_{2,3}(n) + 3g_{2,3}(n) + 3h_{2,3}(n) + t_{2,3}(n)$ . The initial values are the same as for  $SG_2$ :  $f_{2,3}(0) = 1$ ,  $g_{2,3}(0) = 0$ ,  $h_{2,3}(0) = 1$  and  $t_{2,3}(0) = 0$ .

The recursion relations are lengthy and given in the appendix. Some values of  $M_{2,3}(n)$ ,  $f_{2,3}(n)$ ,  $g_{2,3}(n)$ ,  $h_{2,3}(n)$ ,  $t_{2,3}(n)$  are listed in Table III. These numbers grow exponentially, and do not have simple integer factorizations.

The sequence of the ratio defined in Eq. (3.8)  $\{\alpha_{2,3}(n)\}_{n=1}^{\infty}$  increases monotonically and  $\{\gamma_{2,3}(n)\}_{n=1}^{\infty}$  decreases monotonically with  $0 \leq \alpha_{2,3}(n) \leq \gamma_{2,3}(n) \leq 1$ , in contrast to the results for  $SG_2(n)$ . The values of  $\alpha_{2,3}(n)$ ,  $\beta_{2,3}(n)$ ,  $\gamma_{2,3}(n)$  for small  $n$  are listed in Table IV.

By a similar argument as Lemma III.4, the asymptotic growth constant for the number of dimer-monomers on  $SG_{2,3}(n)$  is bounded:

$$\frac{5 \ln f_{2,3}(m) + 6 \ln[1 + 2\alpha_{2,3}(m)] + \ln[1 + 3\alpha_{2,3}(m)]}{7 \times 6^m} \leq z_{SG_{2,3}} \leq \frac{5 \ln f_{2,3}(m) + 6 \ln[1 + 2\gamma_{2,3}(m)] + \ln[1 + 3\gamma_{2,3}(m)]}{7 \times 6^m}, \quad (4.3)$$

TABLE III: The first few values of  $M_{2,3}(n)$ ,  $f_{2,3}(n)$ ,  $g_{2,3}(n)$ ,  $h_{2,3}(n)$ ,  $t_{2,3}(n)$ .

$n$	0	1	2
$M_{2,3}(n)$	4	425	755,290,432,490,932
$f_{2,3}(n)$	1	66	116,464,644,336,176
$g_{2,3}(n)$	0	56	100,722,462,529,064
$h_{2,3}(n)$	1	49	87,108,127,443,640
$t_{2,3}(n)$	0	44	75,334,018,236,644

TABLE IV: The first few values of  $\alpha_{2,3}(n)$ ,  $\beta_{2,3}(n)$ ,  $\gamma_{2,3}(n)$ . The last digits given are rounded off.

$n$	1	2	3
$\alpha_{2,3}(n)$	0.848484848484848	0.864832955126948	0.864833096846111
$\beta_{2,3}(n)$	0.875	0.864833178780796	0.864833096846111
$\gamma_{2,3}(n)$	0.897959183673469	0.864833402432925	0.864833096846111

with  $m$  a positive integer. We have the following proposition.

**Proposition IV.1** *The asymptotic growth constant for the number of dimer-monomers on the generalized two-dimensional Sierpinski gasket  $SG_{2,3}(n)$  in the large  $n$  limit is  $z_{SG_{2,3}} = 0.671617161058\dots$*

The convergence of the upper and lower bounds remains quick. More than a hundred significant figures for  $z_{SG_{2,3}}$  can be obtained when  $m$  in Eq. (4.3) is equal to five.

For  $SG_{2,4}(n)$ , the numbers of edges and vertices are given by

$$e(SG_{2,4}(n)) = 3 \times 10^n, \quad (4.4)$$

$$v(SG_{2,4}(n)) = \frac{4 \times 10^n + 5}{3}, \quad (4.5)$$

where again the three outmost vertices have degree two. There are  $(10^n - 1)/3$  vertices of  $SG_{2,4}(n)$  with degree six, and  $(10^n - 1)$  vertices with degree four. By Definition III.1, the

number of dimer-monomers is  $M_{2,4}(n) = f_{2,4}(n) + 3g_{2,4}(n) + 3h_{2,4}(n) + t_{2,4}(n)$ . The initial values are the same as for  $SG_2$ :  $f_{2,4}(0) = 1$ ,  $g_{2,4}(0) = 0$ ,  $h_{2,4}(0) = 1$  and  $t_{2,4}(0) = 0$ . We write a computer program to obtain the recursion relations for  $SG_{2,4}(n)$ . They are too lengthy to be included here and are available from the authors on request. Some values of  $M_{2,4}(n)$ ,  $f_{2,4}(n)$ ,  $g_{2,4}(n)$ ,  $h_{2,4}(n)$ ,  $t_{2,4}(n)$  are listed in Table V. These numbers grow exponentially, and do not have simple integer factorizations.

TABLE V: The first few values of  $M_{2,4}(n)$ ,  $f_{2,4}(n)$ ,  $g_{2,4}(n)$ ,  $h_{2,4}(n)$ ,  $t_{2,4}(n)$ .

$n$	0	1	2
$M_{2,4}(n)$	4	14,278	7,033,761,314,434,948,243,456,944,474,554,222,281,728
$f_{2,4}(n)$	1	2,220	1,095,249,688,634,151,454,219,516,689,432,826,798,080
$g_{2,4}(n)$	0	1,914	940,563,707,718,765,231,855,988,194,853,818,067,968
$h_{2,4}(n)$	1	1,640	807,724,574,091,886,425,362,687,789,454,449,995,776
$t_{2,4}(n)$	0	1,396	693,646,780,368,841,817,581,399,832,196,591,292,416

The sequence of the ratio defined in Eq. (3.8)  $\{\alpha_{2,4}(n)\}_{n=1}^{\infty}$  decreases monotonically and  $\{\gamma_{2,4}(n)\}_{n=1}^{\infty}$  increases monotonically with  $0 \leq \gamma_{2,4}(n) \leq \alpha_{2,4}(n) \leq 1$ , the same as the results for  $SG_2(n)$ .

The values of  $\alpha_{2,4}(n)$ ,  $\beta_{2,4}(n)$ ,  $\gamma_{2,4}(n)$  for small  $n$  are listed in Table VI.

TABLE VI: The first few values of  $\alpha_{2,4}(n)$ ,  $\beta_{2,4}(n)$ ,  $\gamma_{2,4}(n)$ . The last digits given are rounded off.

$n$	1	2	3
$\alpha_{2,4}(n)$	0.862162162162162	0.858766468942539	0.858766468941692
$\beta_{2,4}(n)$	0.856844305120167	0.858766468941199	0.858766468941692
$\gamma_{2,4}(n)$	0.851219512195122	0.858766468939860	0.858766468941692

By a similar argument as Lemma III.4, the asymptotic growth constant for the number of dimer-monomers on  $SG_{2,4}(n)$  is bounded:

$$\frac{3 \ln f_{2,4}(m) + 3 \ln[1 + 2\gamma_{2,4}(m)] + \ln[1 + 3\gamma_{2,4}(m)]}{4 \times 10^m} \leq z_{SG_{2,4}} \leq \frac{3 \ln f_{2,4}(m) + 3 \ln[1 + 2\alpha_{2,4}(m)] + \ln[1 + 3\alpha_{2,4}(m)]}{4 \times 10^m}, \quad (4.6)$$



with  $m$  a positive integer. We have the following proposition.

**Proposition IV.2** *The asymptotic growth constant for the number of dimer-monomers on the generalized two-dimensional Sierpinski gasket  $SG_{2,4}(n)$  in the large  $n$  limit is  $z_{SG_{2,4}} = 0.684872262332\dots$*

The convergence of the upper and lower bounds is again quick. More than a hundred significant figures for  $z_{SG_{2,4}}$  can be obtained when  $m$  in Eq. (4.6) is equal to four.

## V. THE NUMBER OF DIMER-MONOMERS ON $SG_d(n)$ WITH $d = 3, 4$

In this section, we derive the asymptotic growth constants of dimer-monomers on  $SG_d(n)$  with  $d = 3, 4$ . For the three-dimensional Sierpinski gasket  $SG_3(n)$ , we use the following definitions.

**Definition V.1** *Consider the three-dimensional Sierpinski gasket  $SG_3(n)$  at stage  $n$ . (a) Define  $M_3(n) \equiv N_{DM}(SG_3(n))$  as the number of dimer-monomers. (b) Define  $f_3(n)$  as the number of dimer-monomers such that the four outmost vertices are occupied by monomers. (c) Define  $g_3(n)$  as the number of dimer-monomers such that one of the outmost vertices is occupied by a dimer and the other three outmost vertices are occupied by monomers. (d) Define  $h_3(n)$  as the number of dimer-monomers such that two of the outmost vertices are occupied by monomers and the other two outmost vertices are occupied by dimers. (e) Define  $r_3(n)$  as the number of dimer-monomers such that one of the outmost vertices is occupied by a monomer and the other three outmost vertices are occupied by dimers. (f) Define  $s_3(n)$  as the number of dimer-monomers such that all four outmost vertices are occupied by dimers.*

The quantities  $M_3(n)$ ,  $f_3(n)$ ,  $g_3(n)$ ,  $h_3(n)$ ,  $r_3(n)$  and  $s_3(n)$  are illustrated in Fig. 9, where only the outmost vertices are shown. There are  $\binom{4}{1} = 4$  equivalent  $g_3(n)$ ,  $\binom{4}{2} = 6$  equivalent  $h_3(n)$ , and  $\binom{4}{1} = 4$  equivalent  $r_3(n)$ . By definition,

$$M_3(n) = f_3(n) + 4g_3(n) + 6h_3(n) + 4r_3(n) + s_3(n) . \quad (5.1)$$

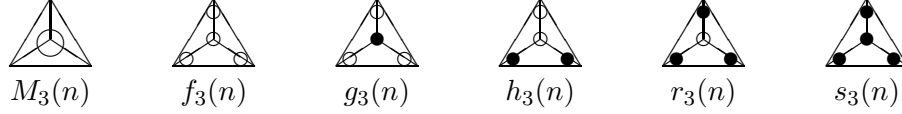


FIG. 9: Illustration for the spanning subgraphs  $M_3(n)$ ,  $f_3(n)$ ,  $g_3(n)$ ,  $h_3(n)$ ,  $r_3(n)$  and  $s_3(n)$ . Only the four outmost vertices are shown explicitly for  $f_3(n)$ ,  $g_3(n)$ ,  $h_3(n)$ ,  $r_3(n)$  and  $s_3(n)$ , where each open circle is occupied by a monomer and each solid circle is occupied by a dimer.

The initial values at stage zero are  $f_3(0) = 1$ ,  $g_3(0) = 0$ ,  $h_3(0) = 1$ ,  $r_3(0) = 0$ ,  $s_3(0) = 3$  and  $M_3(0) = 10$ .

The recursion relations are lengthy and given in the appendix. Some values of  $M_3(n)$ ,  $f_3(n)$ ,  $g_3(n)$ ,  $h_3(n)$ ,  $r_3(n)$ ,  $s_3(n)$  are listed in Table VII. These numbers grow exponentially, and do not have simple integer factorizations.

TABLE VII: The first few values of  $M_3(n)$ ,  $f_3(n)$ ,  $g_3(n)$ ,  $h_3(n)$ ,  $r_3(n)$ ,  $s_3(n)$ .

$n$	0	1	2	3
$M_3(n)$	10	945	132,820,373,046	49,123,375,811,021,432,878,640,796,802,876,545,882,185,505
$f_3(n)$	1	51	7,365,569,811	2,724,928,560,954,289,860,903,291,271,266,882,549,492,483
$g_3(n)$	0	57	7,816,070,424	2,889,924,536,764,017,260,444,663,495,693,780,813,791,233
$h_3(n)$	1	62	8,289,450,499	3,064,910,998,294,837,201,844,707,724,238,032,710,560,958
$r_3(n)$	0	60	8,786,476,992	3,250,492,861,272,219,038,243,497,885,127,347,116,333,900
$s_3(n)$	3	54	9,307,910,577	3,447,311,668,153,174,611,916,613,662,896,955,348,826,742

Define  $\alpha_3(n) = g_3(n)/f_3(n)$  and  $\gamma_3(n) = s_3(n)/r_3(n)$  as in Eq. (3.8). We find  $\{\alpha_3(n)\}_{n=1}^{\infty}$  decreases monotonically and  $\{\gamma_3(n)\}_{n=1}^{\infty}$  increases monotonically with  $1 \leq \gamma_3(n) \leq \alpha_3(n)$  for  $n \geq 2$ . The values of  $\alpha_3(n)$ ,  $\gamma_3(n)$  and other ratios for small  $n$  are listed in Table VIII.

By a similar argument as Lemma III.4, the asymptotic growth constant for the number of dimer-monomers on  $SG_3(n)$  is bounded:

$$\frac{\ln f_3(m) + 2 \ln[1 + 2\gamma_3(m)]}{2 \times 4^m} \leq z_{SG_3} \leq \frac{\ln f_3(m) + 2 \ln[1 + 2\alpha_3(m)]}{2 \times 4^m}, \quad (5.2)$$

TABLE VIII: The first few values of  $\alpha_3(n)$ ,  $\gamma_3(n)$  and other ratios. The last digits given are rounded off.

$n$	1	2	3	4
$\alpha_3(n)$	1.11764705882353	1.06116303620220	1.06055056935215	1.06055052894365
$h_3(n)/g_3(n)$	1.08771929824561	1.06056497054408	1.06055052971271	1.06055052894365
$r_3(n)/h_3(n)$	0.96774193548387	1.05995891923837	1.06055049007316	1.06055052894365
$\gamma_3(n)$	0.9	1.05934501228135	1.06055045043351	1.06055052894365

with  $m$  a positive integer. We have the following proposition.

**Proposition V.1** *The asymptotic growth constant for the number of dimer-monomers on the three-dimensional Sierpinski gasket  $SG_3(n)$  in the large  $n$  limit is  $z_{SG_3} = 0.781151467411\dots$*

The convergence of the upper and lower bounds is as quick as for the ordinary two dimensional case. More than a hundred significant figures for  $z_{SG_3}$  can be obtained when  $m$  in Eq. (5.2) is equal to seven.

For the four-dimensional Sierpinski gasket  $SG_4(n)$ , we use the following definitions.

**Definition V.2** *Consider the four-dimensional Sierpinski gasket  $SG_4(n)$  at stage  $n$ . (a) Define  $M_4(n) \equiv N_{DM}(SG_4(n))$  as the number of dimer-monomers. (b) Define  $f_4(n)$  as the number of dimer-monomers such that the five outmost vertices are occupied by monomers. (c) Define  $g_4(n)$  as the number of dimer-monomers such that one of the outmost vertices is occupied by a dimer and the other four outmost vertices are occupied by monomers. (d) Define  $h_4(n)$  as the number of dimer-monomers such that two of the outmost vertices are occupied by dimers and the other three outmost vertices are occupied by monomers. (e) Define  $r_4(n)$  as the number of dimer-monomers such that two of the outmost vertices are occupied by monomers and the other three outmost vertices are occupied by dimers. (f) Define  $s_4(n)$  as the number of dimer-monomers such that one of the outmost vertices is occupied by a monomer and the other four outmost vertices are occupied by dimers. (g) Define  $t_4(n)$  as the number of dimer-monomers such that all five outmost vertices are occupied by dimers.*

The quantities  $M_4(n)$ ,  $f_4(n)$ ,  $g_4(n)$ ,  $h_4(n)$ ,  $r_4(n)$ ,  $s_4(n)$  and  $t_4(n)$  are illustrated in Fig. 10, where only the outmost vertices are shown. There are  $\binom{5}{1} = 5$  equivalent  $g_4(n)$ ,  $\binom{5}{2} = 10$  equivalent  $h_4(n)$ ,  $\binom{5}{3} = 10$  equivalent  $r_4(n)$ ,  $\binom{5}{1} = 5$  equivalent  $s_4(n)$ . By definition,

$$M_4(n) = f_4(n) + 5g_4(n) + 10h_4(n) + 10r_4(n) + 5s_4(n) + t_4(n) . \quad (5.3)$$

The initial values at stage zero are  $f_4(0) = 1$ ,  $g_4(0) = 0$ ,  $h_4(0) = 1$ ,  $r_4(0) = 0$ ,  $s_4(0) = 3$ ,  $t_4(0) = 0$  and  $M_4(0) = 26$ .

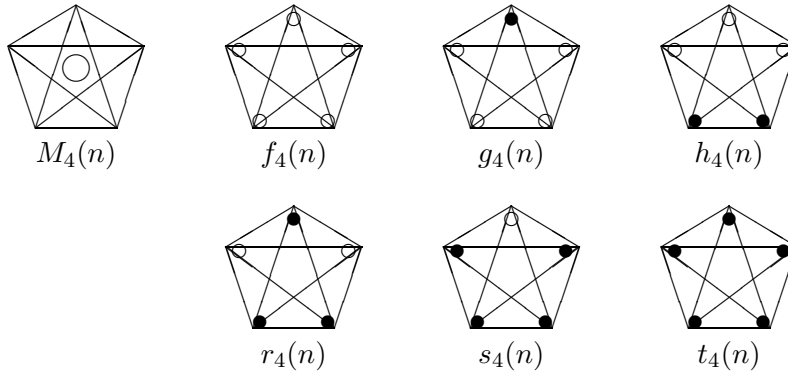


FIG. 10: Illustration for the spanning subgraphs  $M_4(n)$ ,  $f_4(n)$ ,  $g_4(n)$ ,  $h_4(n)$ ,  $r_4(n)$ ,  $s_4(n)$  and  $t_4(n)$ . Only the five outmost vertices are shown explicitly for  $f_4(n)$ ,  $g_4(n)$ ,  $h_4(n)$ ,  $r_4(n)$ ,  $s_4(n)$  and  $t_4(n)$ , where each open circle is occupied by a monomer and each solid circle is occupied by a dimer.

We write a computer program to obtain the recursion relations for  $SG_4(n)$ . They are too lengthy to be included here, and are available from the authors on request. Some values of  $M_4(n)$ ,  $f_4(n)$ ,  $g_4(n)$ ,  $h_4(n)$ ,  $r_4(n)$ ,  $s_4(n)$ ,  $t_4(n)$  are listed in Table IX. These numbers grow exponentially, and do not have simple integer factorizations.

Define  $\alpha_4(n) = g_4(n)/f_4(n)$  and  $\gamma_4(n) = t_4(n)/s_4(n)$  as in Eq. (3.8). We find  $\{\alpha_4(n)\}_{n=1}^{\infty}$  decreases monotonically and  $\{\gamma_4(n)\}_{n=1}^{\infty}$  increases monotonically with  $1 \leq \gamma_4(n) \leq \alpha_4(n)$  for positive integer  $n$ . The values of  $\alpha_4(n)$ ,  $\gamma_4(n)$  and other ratios for small  $n$  are listed in Table X.

TABLE IX: The first few values of  $M_4(n)$ ,  $f_4(n)$ ,  $g_4(n)$ ,  $h_4(n)$ ,  $r_4(n)$ ,  $s_4(n)$ ,  $t_4(n)$ .

$n$	0	1	2
$M_4(n)$	26	141,339	1,567,220,397,434,550,336,692,928
$f_4(n)$	1	2,460	27,951,923,701,499,685,610,752
$g_4(n)$	0	3,168	34,593,006,758,221,606,500,864
$h_4(n)$	1	3,990	42,806,033,106,111,666,338,688
$r_4(n)$	0	4,852	52,961,649,817,161,203,920,896
$s_4(n)$	3	5,683	65,517,552,720,775,495,239,744
$t_4(n)$	0	6,204	81,038,847,105,336,439,783,296

TABLE X: The first few values of  $\alpha_4(n)$ ,  $\gamma_4(n)$  and other ratios. The last digits given are rounded off.

$n$	1	2	3	4
$\alpha_4(n)$	1.28780487804878	1.23758948141253	1.23734576161280	1.23734575732423
$h_4(n)/g_4(n)$	1.25946969696970	1.23741869000555	1.23734575860203	1.23734575732423
$r_4(n)/h_4(n)$	1.21604010025063	1.23724732179398	1.23734575559125	1.23734575732423
$s_4(n)/r_4(n)$	1.17126957955482	1.23707537334960	1.23734575258048	1.23734575732423
$\gamma_4(n)$	1.09167693119831	1.23690284114717	1.23734574956971	1.23734575732423

By a similar argument as Lemma III.4, the asymptotic growth constant for the number of dimer-monomers on  $SG_4(n)$  is bounded:

$$\frac{2 \ln f_4(m) + 5 \ln[1 + 2\gamma_4(m)]}{5^{m+1}} \leq z_{SG_4} \leq \frac{2 \ln f_4(m) + 5 \ln[1 + 2\alpha_4(m)]}{5^{m+1}}, \quad (5.4)$$

with  $m$  a positive integer. We have the following proposition.

**Proposition V.2** *The asymptotic growth constant for the number of dimer-monomers on the four-dimensional Sierpinski gasket  $SG_4(n)$  in the large  $n$  limit is  $z_{SG_4} = 0.876779402949\dots$*

The convergence of the upper and lower bounds is as quick as for the ordinary two dimensional case. More than a hundred significant figures for  $z_{SG_4}$  can be obtained when  $m$  in Eq. (5.4) is equal to seven.

## VI. SUMMARY

The bounds of the asymptotic growth constants for dimer-monomers on  $SG_2(n)$ ,  $SG_3(n)$  and  $SG_4(n)$  given in sections III and V lead to the following conjecture for general  $SG_d(n)$ .

**Conjecture VI.1** *Define  $\alpha_d(n)$  as the ratio: the number of dimer-monomers on  $SG_d(n)$  with all but one outmost vertices covered by monomers divided by that with all outmost vertices covered by monomers; define  $\gamma_d(n)$  as the ratio: the number of dimer-monomers on  $SG_d(n)$  with all outmost vertices covered by dimers divided by that with all but one outmost vertices covered by dimers. The asymptotic growth constant for the number of dimer-monomers on the  $d$ -dimensional Sierpinski gasket  $SG_d$  is bounded*

$$\frac{2 \ln f_d(m) + (d+1) \ln[1 + 2\gamma_d(m)]}{(d+1)^{m+1}} \leq z_{SG_d} \leq \frac{2 \ln f_d(m) + (d+1) \ln[1 + 2\alpha_d(m)]}{(d+1)^{m+1}}. \quad (6.1)$$

We notice that the convergence of the upper and lower bounds of the asymptotic growth constants for dimer-monomers on  $SG_d(n)$  is about the same for each integer  $d \geq 2$ , in contrast to the results observed in [35] for spanning forests on  $SG_d(n)$  where the convergence of the bounds of the asymptotic growth constants becomes slow when  $d$  increases.

The values of  $z_{SG_d}$  increases as dimension  $d$  increases. Similarly for the generalized two-dimensional Sierpinski gasket, the values of  $z_{SG_{2,b}}$  increases slightly as  $b$  increases.

Compare the present results with those in Ref. [36], we find that the number of dimer-monomers on the Sierpinski gasket  $SG_d(n)$  is less than that of spanning trees in general.

## APPENDIX A: RECURSION RELATIONS FOR $SG_{2,3}(n)$

We give the recursion relations for the generalized two-dimensional Sierpinski gasket  $SG_{2,3}(n)$  here. For any non-negative integer  $n$ , we have

$$f_{2,3}(n+1)$$

$$\begin{aligned}
&= f_{2,3}^6(n) + 15f_{2,3}^5(n)g_{2,3}(n) + 12f_{2,3}^5(n)h_{2,3}(n) + 84f_{2,3}^4(n)g_{2,3}^2(n) + 3f_{2,3}^5(n)t_{2,3}(n) \\
&\quad + 117f_{2,3}^4(n)g_{2,3}(n)h_{2,3}(n) + 220f_{2,3}^3(n)g_{2,3}^3(n) + 24f_{2,3}^4(n)g_{2,3}(n)t_{2,3}(n) + 33f_{2,3}^4(n)h_{2,3}^2(n) \\
&\quad + 390f_{2,3}^3(n)g_{2,3}^2(n)h_{2,3}(n) + 273f_{2,3}^2(n)g_{2,3}^4(n) + 9f_{2,3}^4(n)h_{2,3}(n)t_{2,3}(n) \\
&\quad + 63f_{2,3}^3(n)g_{2,3}^2(n)t_{2,3}(n) + 180f_{2,3}^3(n)g_{2,3}(n)h_{2,3}^2(n) + 519f_{2,3}^2(n)g_{2,3}^3(n)h_{2,3}(n) \\
&\quad + 141f_{2,3}(n)g_{2,3}^5(n) + 36f_{2,3}^3(n)g_{2,3}(n)h_{2,3}(n)t_{2,3}(n) + 60f_{2,3}^2(n)g_{2,3}^3(n)t_{2,3}(n) \\
&\quad + 20f_{2,3}^3(n)h_{2,3}^3(n) + 264f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}^2(n) + 240f_{2,3}(n)g_{2,3}^4(n)h_{2,3}(n) + 20g_{2,3}^6(n) \\
&\quad + 3f_{2,3}^3(n)h_{2,3}^2(n)t_{2,3}(n) + 30f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}(n)t_{2,3}(n) + 15f_{2,3}(n)g_{2,3}^4(n)t_{2,3}(n) \\
&\quad + 33f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^3(n) + 90f_{2,3}(n)g_{2,3}^3(n)h_{2,3}^2(n) + 21g_{2,3}^5(n)h_{2,3}(n) , \tag{A1}
\end{aligned}$$

$$g_{2,3}(n+1)$$

$$\begin{aligned}
&= f_{2,3}^5(n)g_{2,3}(n) + 2f_{2,3}^5(n)h_{2,3}(n) + 13f_{2,3}^4(n)g_{2,3}^2(n) + f_{2,3}^5(n)t_{2,3}(n) \\
&\quad + 35f_{2,3}^4(n)g_{2,3}(n)h_{2,3}(n) + 60f_{2,3}^3(n)g_{2,3}^3(n) + 14f_{2,3}^4(n)g_{2,3}(n)t_{2,3}(n) + 18f_{2,3}^4(n)h_{2,3}^2(n) \\
&\quad + 188f_{2,3}^3(n)g_{2,3}^2(n)h_{2,3}(n) + 120f_{2,3}^2(n)g_{2,3}^4(n) + 11f_{2,3}^4(n)h_{2,3}(n)t_{2,3}(n) \\
&\quad + 61f_{2,3}^3(n)g_{2,3}^2(n)t_{2,3}(n) + 152f_{2,3}^3(n)g_{2,3}(n)h_{2,3}^2(n) + 397f_{2,3}^2(n)g_{2,3}^3(n)h_{2,3}(n) \\
&\quad + 99f_{2,3}(n)g_{2,3}^5(n) + f_{2,3}^4(n)t_{2,3}^2(n) + 72f_{2,3}^3(n)g_{2,3}(n)h_{2,3}(n)t_{2,3}(n) \\
&\quad + 102f_{2,3}^2(n)g_{2,3}^3(n)t_{2,3}(n) + 30f_{2,3}^3(n)h_{2,3}^3(n) + 372f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}^2(n) \\
&\quad + 310f_{2,3}(n)g_{2,3}^4(n)h_{2,3}(n) + 25g_{2,3}^6(n) + 4f_{2,3}^3(n)g_{2,3}(n)t_{2,3}^2(n) + 13f_{2,3}^3(n)h_{2,3}^2(n)t_{2,3}(n)
\end{aligned}$$

$$\begin{aligned}
& +130f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}(n)t_{2,3}(n) + 57f_{2,3}(n)g_{2,3}^4(n)t_{2,3}(n) + 107f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^3(n) \\
& +266f_{2,3}(n)g_{2,3}^3(n)h_{2,3}^2(n) + 63g_{2,3}^5(n)h_{2,3}(n) + 4f_{2,3}^2(n)g_{2,3}^2(n)t_{2,3}^2(n) \\
& +30f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^2(n)t_{2,3}(n) + 52f_{2,3}(n)g_{2,3}^3(n)h_{2,3}(n)t_{2,3}(n) + 6g_{2,3}^5(n)t_{2,3}(n) \\
& +6f_{2,3}^2(n)h_{2,3}^4(n) + 60f_{2,3}(n)g_{2,3}^2(n)h_{2,3}^3(n) + 34g_{2,3}^4(n)h_{2,3}^2(n) , \tag{A2}
\end{aligned}$$

$$h_{2,3}(n+1)$$

$$\begin{aligned}
= & f_{2,3}^4(n)g_{2,3}^2(n) + 4f_{2,3}^4(n)g_{2,3}(n)h_{2,3}(n) + 11f_{2,3}^3(n)g_{2,3}^3(n) + 2f_{2,3}^4(n)g_{2,3}(n)t_{2,3}(n) \\
& +4f_{2,3}^4(n)h_{2,3}^2(n) + 50f_{2,3}^3(n)g_{2,3}^2(n)h_{2,3}(n) + 40f_{2,3}^2(n)g_{2,3}^4(n) + 4f_{2,3}^4(n)h_{2,3}(n)t_{2,3}(n) \\
& +21f_{2,3}^3(n)g_{2,3}^2(n)t_{2,3}(n) + 68f_{2,3}^3(n)g_{2,3}(n)h_{2,3}^2(n) + 191f_{2,3}^2(n)g_{2,3}^3(n)h_{2,3}(n) \\
& +56f_{2,3}(n)g_{2,3}^5(n) + f_{2,3}^4(n)t_{2,3}^2(n) + 52f_{2,3}^3(n)g_{2,3}(n)h_{2,3}(n)t_{2,3}(n) \\
& +66f_{2,3}^2(n)g_{2,3}^3(n)t_{2,3}(n) + 25f_{2,3}^3(n)h_{2,3}^3(n) + 289f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}^2(n) \\
& +263f_{2,3}(n)g_{2,3}^4(n)h_{2,3}(n) + 24g_{2,3}^6(n) + 9f_{2,3}^3(n)g_{2,3}(n)t_{2,3}^2(n) + 23f_{2,3}^3(n)h_{2,3}^2(n)t_{2,3}(n) \\
& +167f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}(n)t_{2,3}(n) + 72f_{2,3}(n)g_{2,3}^4(n)t_{2,3}(n) + 150f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^3(n) \\
& +390f_{2,3}(n)g_{2,3}^3(n)h_{2,3}^2(n) + 101g_{2,3}^5(n)h_{2,3}(n) + 5f_{2,3}^3(n)h_{2,3}(n)t_{2,3}^2(n) \\
& +19f_{2,3}^2(n)g_{2,3}^2(n)t_{2,3}^2(n) + 94f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^2(n)t_{2,3}(n) + 158f_{2,3}(n)g_{2,3}^3(n)h_{2,3}(n)t_{2,3}(n) \\
& +20g_{2,3}^5(n)t_{2,3}(n) + 20f_{2,3}^2(n)h_{2,3}^4(n) + 201f_{2,3}(n)g_{2,3}^2(n)h_{2,3}^3(n) + 123g_{2,3}^4(n)h_{2,3}^2(n) \\
& +11f_{2,3}^2(n)g_{2,3}(n)h_{2,3}(n)t_{2,3}^2(n) + 9f_{2,3}(n)g_{2,3}^3(n)t_{2,3}^2(n) + 9f_{2,3}^2(n)h_{2,3}^3(n)t_{2,3}(n)
\end{aligned}$$



$$\begin{aligned}
& +68f_{2,3}(n)g_{2,3}^2(n)h_{2,3}^2(n)t_{2,3}(n) + 27g_{2,3}^4(n)h_{2,3}(n)t_{2,3}(n) + 27f_{2,3}(n)g_{2,3}(n)h_{2,3}^4(n) \\
& +41g_{2,3}^3(n)h_{2,3}^3(n) , \tag{A3}
\end{aligned}$$

$$t_{2,3}(n+1)$$

$$\begin{aligned}
= & f_{2,3}^3(n)g_{2,3}^3(n) + 6f_{2,3}^3(n)g_{2,3}^2(n)h_{2,3}(n) + 9f_{2,3}^2(n)g_{2,3}^4(n) + 3f_{2,3}^3(n)g_{2,3}^2(n)t_{2,3}(n) \\
& +12f_{2,3}^3(n)g_{2,3}(n)h_{2,3}^2(n) + 57f_{2,3}^2(n)g_{2,3}^3(n)h_{2,3}(n) + 24f_{2,3}(n)g_{2,3}^5(n) \\
& +12f_{2,3}^3(n)g_{2,3}(n)h_{2,3}(n)t_{2,3}(n) + 24f_{2,3}^2(n)g_{2,3}^3(n)t_{2,3}(n) + 8f_{2,3}^3(n)h_{2,3}^3(n) \\
& +126f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}^2(n) + 150f_{2,3}(n)g_{2,3}^4(n)h_{2,3}(n) + 20g_{2,3}^6(n) + 3f_{2,3}^3(n)g_{2,3}(n)t_{2,3}^2(n) \\
& +12f_{2,3}^3(n)h_{2,3}^2(n)t_{2,3}(n) + 99f_{2,3}^2(n)g_{2,3}^2(n)h_{2,3}(n)t_{2,3}(n) + 51f_{2,3}(n)g_{2,3}^4(n)t_{2,3}(n) \\
& +111f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^3(n) + 324f_{2,3}(n)g_{2,3}^3(n)h_{2,3}^2(n) + 120g_{2,3}^5(n)h_{2,3}(n) \\
& +6f_{2,3}^3(n)h_{2,3}(n)t_{2,3}^2(n) + 18f_{2,3}^2(n)g_{2,3}^2(n)t_{2,3}^2(n) + 117f_{2,3}^2(n)g_{2,3}(n)h_{2,3}^2(n)t_{2,3}(n) \\
& +186f_{2,3}(n)g_{2,3}^3(n)h_{2,3}(n)t_{2,3}(n) + 33g_{2,3}^5(n)t_{2,3}(n) + 30f_{2,3}^2(n)h_{2,3}^4(n) \\
& +282f_{2,3}(n)g_{2,3}^2(n)h_{2,3}^3(n) + 240g_{2,3}^4(n)h_{2,3}^2(n) + f_{2,3}^3(n)t_{2,3}^3(n) \\
& +36f_{2,3}^2(n)g_{2,3}(n)h_{2,3}(n)t_{2,3}^2(n) + 21f_{2,3}(n)g_{2,3}^3(n)t_{2,3}^2(n) + 33f_{2,3}^2(n)h_{2,3}^3(n)t_{2,3}(n) \\
& +183f_{2,3}(n)g_{2,3}^2(n)h_{2,3}^2(n)t_{2,3}(n) + 99g_{2,3}^4(n)h_{2,3}(n)t_{2,3}(n) + 87f_{2,3}(n)g_{2,3}(n)h_{2,3}^4(n) \\
& +180g_{2,3}^3(n)h_{2,3}^3(n) + 3f_{2,3}^2(n)g_{2,3}(n)t_{2,3}^3(n) + 9f_{2,3}^2(n)h_{2,3}^2(n)t_{2,3}^2(n) \\
& +24f_{2,3}(n)g_{2,3}^2(n)h_{2,3}(n)t_{2,3}^2(n) + 6g_{2,3}^4(n)t_{2,3}^2(n) + 42f_{2,3}(n)g_{2,3}(n)h_{2,3}^3(n)t_{2,3}(n)
\end{aligned}$$

$$+63g_{2,3}^3(n)h_{2,3}^2(n)t_{2,3}(n) + 6f_{2,3}(n)h_{2,3}^5(n) + 39g_{2,3}^2(n)h_{2,3}^4(n) . \quad (\text{A4})$$

There are always  $2916 = 4 \times 3^6$  terms in these equations.

## APPENDIX B: RECURSION RELATIONS FOR $SG_3(n)$

We give the recursion relations for the three-dimensional Sierpinski gasket  $SG_3(n)$  here. For any non-negative integer  $n$ , we have

$$\begin{aligned} & f_3(n+1) \\ &= f_3^4(n) + 12f_3^3(n)g_3(n) + 12f_3^3(n)h_3(n) + 48f_3^2(n)g_3^2(n) + 4f_3^3(n)r_3(n) \\ & \quad + 84f_3^2(n)g_3(n)h_3(n) + 72f_3(n)g_3^3(n) + 24f_3^2(n)g_3(n)r_3(n) + 30f_3^2(n)h_3^2(n) \\ & \quad + 156f_3(n)g_3^2(n)h_3(n) + 30g_3^4(n) + 12f_3^2(n)h_3(n)r_3(n) + 36f_3(n)g_3^2(n)r_3(n) \\ & \quad + 84f_3(n)g_3(n)h_3^2(n) + 60g_3^3(n)h_3(n) + 24f_3(n)g_3(n)h_3(n)r_3(n) + 8g_3^3(n)r_3(n) \\ & \quad + 8f_3(n)h_3^3(n) + 24g_3^2(n)h_3^2(n) , \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} & g_3(n+1) \\ &= f_3^3(n)g_3(n) + 3f_3^3(n)h_3(n) + 9f_3^2(n)g_3^2(n) + 3f_3^3(n)r_3(n) + 33f_3^2(n)g_3(n)h_3(n) \\ & \quad + 24f_3(n)g_3^3(n) + f_3^3(n)s_3(n) + 24f_3^2(n)g_3(n)r_3(n) + 21f_3^2(n)h_3^2(n) + 96f_3(n)g_3^2(n)h_3(n) \\ & \quad + 18g_3^4(n) + 6f_3^2(n)g_3(n)s_3(n) + 21f_3^2(n)h_3(n)r_3(n) + 51f_3(n)g_3^2(n)r_3(n) \\ & \quad + 93f_3(n)g_3(n)h_3^2(n) + 69g_3^3(n)h_3(n) + 3f_3^2(n)h_3(n)s_3(n) + 9f_3(n)g_3^2(n)s_3(n) \\ & \quad + 3f_3^2(n)r_3^2(n) + 66f_3(n)g_3(n)h_3(n)r_3(n) + 24g_3^3(n)r_3(n) + 21f_3(n)h_3^3(n) + 66g_3^2(n)h_3^2(n) \end{aligned}$$

$$\begin{aligned}
& +6f_3(n)g_3(n)h_3(n)s_3(n) + 2g_3^3(n)s_3(n) + 6f_3(n)g_3(n)r_3^2(n) + 12f_3(n)h_3^2(n)r_3(n) \\
& +24g_3^2(n)h_3(n)r_3(n) + 14g_3(n)h_3^3(n) , \tag{B2}
\end{aligned}$$

$$h_3(n+1)$$

$$\begin{aligned}
= & f_3^2(n)g_3^2(n) + 6f_3^2(n)g_3(n)h_3(n) + 6f_3(n)g_3^3(n) + 6f_3^2(n)g_3(n)r_3(n) + 8f_3^2(n)h_3^2(n) \\
& +38f_3(n)g_3^2(n)h_3(n) + 8g_3^4(n) + 2f_3^2(n)g_3(n)s_3(n) + 14f_3^2(n)h_3(n)r_3(n) \\
& +30f_3(n)g_3^2(n)r_3(n) + 64f_3(n)g_3(n)h_3^2(n) + 50g_3^3(n)h_3(n) + 4f_3^2(n)h_3(n)s_3(n) \\
& +8f_3(n)g_3^2(n)s_3(n) + 5f_3^2(n)r_3^2(n) + 80f_3(n)g_3(n)h_3(n)r_3(n) + 30g_3^3(n)r_3(n) \\
& +26f_3(n)h_3^3(n) + 87g_3^2(n)h_3^2(n) + 2f_3^2(n)r_3(n)s_3(n) + 16f_3(n)g_3(n)h_3(n)s_3(n) \\
& +6g_3^3(n)s_3(n) + 18f_3(n)g_3(n)r_3^2(n) + 34f_3(n)h_3^2(n)r_3(n) + 72g_3^2(n)h_3(n)r_3(n) \\
& +44g_3(n)h_3^3(n) + 4f_3(n)g_3(n)r_3(n)s_3(n) + 4f_3(n)h_3^2(n)s_3(n) + 8g_3^2(n)h_3(n)s_3(n) \\
& +8f_3(n)h_3(n)r_3^2(n) + 8g_3^2(n)r_3^2(n) + 28g_3(n)h_3^2(n)r_3(n) + 4h_3^4(n) , \tag{B3}
\end{aligned}$$

$$r_3(n+1)$$

$$\begin{aligned}
= & f_3(n)g_3^3(n) + 9f_3(n)g_3^2(n)h_3(n) + 3g_3^4(n) + 9f_3(n)g_3^2(n)r_3(n) + 24f_3(n)g_3(n)h_3^2(n) \\
& +27g_3^3(n)h_3(n) + 3f_3(n)g_3^2(n)s_3(n) + 42f_3(n)g_3(n)h_3(n)r_3(n) + 22g_3^3(n)r_3(n) \\
& +18f_3(n)h_3^3(n) + 75g_3^2(n)h_3^2(n) + 12f_3(n)g_3(n)h_3(n)s_3(n) + 6g_3^3(n)s_3(n) \\
& +15f_3(n)g_3(n)r_3^2(n) + 39f_3(n)h_3^2(n)r_3(n) + 99g_3^2(n)h_3(n)r_3(n) + 69g_3(n)h_3^3(n)
\end{aligned}$$

$$\begin{aligned}
& +6f_3(n)g_3(n)r_3(n)s_3(n) + 9f_3(n)h_3^2(n)s_3(n) + 21g_3^2(n)h_3(n)s_3(n) + 21f_3(n)h_3(n)r_3^2(n) \\
& +24g_3^2(n)r_3^2(n) + 96g_3(n)h_3^2(n)r_3(n) + 15h_3^4(n) + 6f_3(n)h_3(n)r_3(n)s_3(n) \\
& +6g_3^2(n)r_3(n)s_3(n) + 12g_3(n)h_3^2(n)s_3(n) + 2f_3(n)r_3^3(n) + 24g_3(n)h_3(n)r_3^2(n) \\
& +14h_3^3(n)r_3(n) , \tag{B4}
\end{aligned}$$

$$\begin{aligned}
& s_3(n+1) \\
& = g_3^4(n) + 12g_3^3(n)h_3(n) + 12g_3^3(n)r_3(n) + 48g_3^2(n)h_3^2(n) + 4g_3^3(n)s_3(n) \\
& +84g_3^2(n)h_3(n)r_3(n) + 72g_3(n)h_3^3(n) + 24g_3^2(n)h_3(n)s_3(n) + 30g_3^2(n)r_3^2(n) \\
& +156g_3(n)h_3^2(n)r_3(n) + 30h_3^4(n) + 12g_3^2(n)r_3(n)s_3(n) + 36g_3(n)h_3^2(n)s_3(n) \\
& +84g_3(n)h_3(n)r_3^2(n) + 60h_3^3(n)r_3(n) + 24g_3(n)h_3(n)r_3(n)s_3(n) + 8h_3^3(n)s_3(n) \\
& +8g_3(n)r_3^3(n) + 24h_3^2(n)r_3^2(n) . \tag{B5}
\end{aligned}$$

There are always  $729 = 3^6$  terms in these equations.

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